[12]P. S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981); S. Novak and R. Prehlich, preprint 1981 (unpublished).
[13]Lord Raleigh, Philos. Mat. Ser. 5, **15**, (April 1883).
[14]J. H. Vincent, Proc. Phys. Soc. London **32**, Part II, (February 1919).
[15]H. G. Möller, Z. Telegraph. Telephon. **17**, (April 1921).
[16]E. V. Appleton, Proc. Cambridge Philos. Soc. **21**, (Nov. 1922).
[17]B. Van der Pol, Philos. Mag. **3**, 65 (1927).
[18]M. L. Cartwright, J. Inst. Elec. Eng. **95** III, 88 (1948); Ann. Math. Stud. No. **20**, 202 (1950).
[19]W. C. Lindsey, Proc. IEEE **57**, 1705 (1969).
[20]M. Büttiker and R. Lanclauer, in *Nonlinear Phenomena at Phase Transitions and Instabilities*, edited by T. Riste (Plenum, New York, 1982), p. 111.
[21]A. R. Bishop and S. E. Trullinger, Phys. Rev. B **17**, 2175 (1978).
[22]Reference 3, p. 236–258.
[23]H. A. Kramers, Physica **7**, 284 (1940).
[24]M. Abramowitz and I. A. Stegun, *Handbook of Math. Functions* (Dover, New York 1970), Relation 9.6.16.

Aristotle is not dead: Student understanding of trajectory motion

Robert J. Whitaker
Department of Physics, Southwest Missouri State University, Springfield, Missouri 65802

(Received 1 March 1982; accepted for publication 4 June 1982)

The study of motion presents problems to students of beginning physics at all levels. Part of this problem arises because of students' "common sense" ideas of motion. Six questions on trajectory motion, based on Galileo's "thought experiments" in his *Dialogue Concerning the Two Chief World Systems*, were given to students in several different introductory physics classes. These responses are analyzed and discussed.

INTRODUCTION

Kinematics is one of the first topics encountered by a student beginning his study of physics. It is also a topic with which a student has had limited first-hand experience. A student's interpretation of these experiences, however, is very often different from the ways of understanding motion that were developed by Galileo and Newton and are explained in the textbooks. Indeed, students' understandings of motion are more often Aristotelian than Newtonian.

Historian of science E. J. Dijksterhuis has emphasized the difficulties in understanding the concept of motion. Following an extensive discussion of Aristotelian dynamics, he notes the long period of time necessary to change the Aristotelian concept of motion and observes: "To this day every student of elementary physics has to struggle with the same errors and misconceptions which then had to be overcome, and on a reduced scale,...history repeats itself every year."[1]

The difficulties that students face when encountering simple trajectory motion provide an appropriate example of the deeply ingrained "common sense" approach to their understanding of motion. Anyone who has asked for an explanation of the classic "hunter and monkey" problem on an examination has encountered some of these kinds of responses.

In order to obtain a more complete survey of students' understanding of trajectory motion, students in two sections of a calculus based and two sections of a trigonometry based introductory physics course were asked to respond to six questions prior to beginning their study of motion. These questions dealt with freely falling objects and objects moving along a trajectory. (See Appendix for a list of the questions.) The questions were based, in part, on Galileo's "thought experiments" described in his *Dialogue Concerning the Two Chief World Systems*.[2]

Question 1 compares two freely falling objects of different masses having no horizontal component of velocity. Questions 2, 5, and 6 are different statements of the same problem: an object falls in a reference system moving with a constant horizontal velocity. Questions 3 and 4 are also related problems; one object falls freely from rest, and a second object is given a horizontal velocity at the time of release of the first.

Table I. Responses of students who had taken high school physics ($N = 40$).

Students enrolled in a noncalculus physics course				
Answer	A	B	C	No answer
Question				
1	1 (7%)	0 (0%)	13 (93%)[a]	0 (0%)
2	0 (0%)	11 (79%)[a]	2 (14%)	1 (7%)
3	4 (29%)	10 (71%)[a]	0 (0%)	0 (0%)
4	7 (50%)	7 (50%)[a]	0 (0%)	0 (0%)
5	5 (36%)	6 (43%)[a]	2 (14%)	1 (7%)
6	7 (50%)[a]	6 (43%)	0 (0%)	1 (7%)
Students enrolled in a calculus physics course				
Answer	A	B	C	No answer
Question				
1	3 (12%)	0 (0%)	23 (88%)[a]	0 (0%)
2	1 (4%)	14 (54%)[a]	11 (42%)	0 (0%)
3	8 (31%)	18 (69%)[a]	0 (0%)	0 (0%)
4	6 (23%)	20 (77%)[a]	0 (0%)	0 (0%)
5	6 (23%)	13 (50%)[a]	6 (23%)	1 (4%)
6	12 (46%)[a]	12 (46%)	0 (0%)	2 (8%)

[a] "Correct" responses.

Table II. Responses of students who had not taken high school physics ($N = 60$).

Students enrolled in a non-calculus physics course					
Answer	A	B	C	No answer	More information needed
Question					
1	11 (25%)	0 (0%)	33 (75%)[a]	0 (0%)	0 (0%)
2	0 (0%)	28 (64%)[a]	15 (34%)	1 (2%)	0 (0%)
3	15 (34%)	23 (52%)[a]	6 (14%)	0 (0%)	0 (0%)
4	12 (27%)	25 (57%)[a]	1 (2%)	1 (2%)	5 (11%)
5	8 (18%)	10 (23%)[a]	25 (57%)	1 (2%)	0 (0%)
6	16 (36%)[a]	25 (57%)	1 (2%)	2 (5%)	0 (0%)

Students enrolled in a calculus physics course					
Answer	A	B	C	No answer	More information needed
Question					
1	7 (44%)	0 (0%)	9 (56%)[a]	0 (0%)	0 (0%)
2	0 (0%)	7 (44%)[a]	9 (56%)	0 (0%)	0 (0%)
3	11 (69%)	4 (25%)[a]	1 (6%)	0 (0%)	0 (0%)
4	5 (31%)	9 (56%)[a]	0 (0%)	0 (0%)	2 (12%)
5	2 (12%)	4 (25%)[a]	10 (62%)	0 (0%)	0 (0%)
6	4 (25%)[a]	12 (75%)	0 (0%)	0 (0%)	0 (0%)

[a] "Correct" responses.

The data collected are summarized in Tables I and II. Table I lists the responses of those students who indicated they had taken high school physics. Table II lists the responses of those students who had not taken high school physics. Students who stated they had taken a college physics course, which covered the topic of motion, were excluded from the data analysis.

As noted above, questions 2, 5, and 6 and questions 3 and 4 dealt with similar concepts. Table III summarizes the responses of all students to these two groups of questions. All five questions were correctly answered by only 16 of the respondents. The most revealing responses came from those students who answered a given question correctly and provided a correct reason, but who did not consistently answer a similar question correctly or give the same reason for his choice.

More important than the answers to the questions, however, were the reasons given for that answer. Several reasons given for "correct" answers indicate incomplete or incorrect understanding of the concept involved. Following are representative reasons given for the answer selected.

Table III. Number of responses to similar questions ($N = 100$).

Question No.	2, 5, 6	3, 4
All correct	25	37
One or two correct	41	40
None correct	34	23

SELECTED REASONS FOR ANSWERS TO QUESTIONS

1. A. Because it's heavier.
   The heavier ball will travel faster than the light ball.
   The mass is greater, therefore it will accelerate faster.
   There will be more gravitational pull on the 50-lb ball.
   C. I remember the famous experiment in the past and, more recently, on the moon.
   Any object no matter the mass will fall at 32 ft/s^2.
   They both have the same gravitational pull.
   Gravity acts the same on all objects.
   More force (of gravity) is required to accelerate the heavier ball than the small one. The effect is that they both fall at the same velocity.
2. A. Because the object always falls on a parabolic line.
   B. The bolt is also traveling 60 miles/h.
   The bolt is moving at the same horizontal velocity as the hole.
   Experience with moving objects.
   Zero relative motion.
   The bolt is traveling at the same forward speed as the train; the distance it has to fall is short enough that it won't be significantly affected by gravity.
   C. The train is moving forward and will force the bolt back some.
   The train is accelerating forward and the bolt will have no forward acceleration.
   Since the train is in motion forward and the bolt will only be affected by gravity, the bolt will hit in back of the hole.
   While the bolt is in the air, the train is moving forward.
   The bolt decreases in speed the second it leaves the ceiling while the boxcar (with hole) maintains the speed.
3. A. The centrifical [sic] force of the bullet will defy the gravity longer.
   The bullet will gradually drop; the ball just goes straight down.
   The ball will fall a direct (shorter) distance to the ground, while the bullet will travel a certain distance forward.
   The ball is going straight to the ground while the bullet is going forward and then down.
   The bullet is propelled and the ball is reacting to gravity.
   If there is some theory that operates on the bullet to make it fall at the same rate, I've never heard of it, though it seems possible.
   B. The bullet is falling at the same speed downward as the heavy ball.
   The ever increasing downward fall is common to both of the objects.
   The difference in velocity and the pull of gravity.
   Horizontal movement does not change the effects of gravity.
   Bullet and ball have equal gravity or the vertical velocities will be equal at any time.
   Studied problem in high school.
   C. The bullet is moving faster.
4. A. The bullet has to travel the distance from the gun to the tree, but the monkey will just drop to the ground.
   The monkey is going down away from the target spot of the shot.

Even though the bullet travels faster than the monkey, the monkey only had to move a little bit.
It takes time to reach the monkey. By that time, the monkey fell a distance. [Student correctly derived formula for time of fall in Question 3.]
Experiences with game animals and high school physics.
B. Acceleration due to gravity acts independently of horizontal motion (same as 3).
Gravity that affects the monkey affects the bullet the same way.
Depends on speed of bullet.
The bullet has a parabolic path.
The bullet travels faster than the monkey falls.
Horizontal motion is independent of vertical velocity.
Bullet speed is faster than reaction time of monkey.
Studied similar problem in high school.

5. A. It has only vertical acceleration, no horizontal velocity or acceleration or force exerted on the fall.
Gravity pulls down; this is all that is acting on the ball.
Because of gravity.
Experience and high school physics.
B. Similar to Question 2.
Same as problems 3 and 4. Why so stuck on Galileo and Newton?
The ball will continue its forward motion (like a bomb from a plane).
The forward motion of the rider will carry the ball forward.
Newton's Laws.
C. Because the ball was traveling horizontally, it will still travel horizontally although not as fast as the horse and rider.
Slightly behind the horse and rider. The ball has lost its momentum and is pulled downward while the horse and rider continue their momentum.
In front of the point where it was dropped because of acceleration.

6. A. The ball is moving at the same forward speed as the ship.
Same as all the above. [Which were answered correctly.]
Motionless relative to each other.
The ball has the same momentum forward as the boat.
B. Same general answers as 2 and 5. [Which were answered incorrectly.]
The force of the moving boat will make the ball fall behind the mast.
While the ball is falling, the boat is moving forward.
It still has some horizontal velocity but not as much as the boat.
The boat has greater acceleration.
Once the ball is dropped there is no further force being applied in a forward direction.
C. Motion carries the ball in front of the mast.

DISCUSSION

Fundamental to Aristotelian dynamics is the idea that a force is required *to maintain* an object in motion. Projectile motion presented a particular problem to the Aristotelian model. The vertical motion of a projectile was easily explained by the effect of the weight of the object pulling it downward. The fact of horizontal motion could be explained by assuming that the force given to the object by whatever projected it was maintained, at least briefly, by the object. Depending on the view of the writer, this force was either maintained by the object over its path or was used up quickly and the object simply fell vertically.[3] It is this latter view that Galileo describes in a series of examples in the *Dialogue* to illustrate the Aristotelian viewpoint. Thus Salviati, the voice of Galileo, leads the Aristotelian, Simplicio, into agreeing that a lead ball dropped from the mast of a moving ship "...will strike at that distance from the foot of the mast which the boat will have run during that time of fall of the lead, and for no other reason than that the natural movement of the ball when set free is in a straight line toward the center of the earth."[4] Salviati then proceeds to demonstrate that this explanation is not correct and that the ball "conserves" its horizontal motion and moves along with the ship, falling at the foot of the mast.

Also explicit in the explanation of free fall developed by Aristotle and later medieval writers is the role of friction of the medium through which the body is falling. Thus, while the speed of an object is directly proportional to its weight, it is also inversely proportional to the resistance of the medium. This proportionality holds true until the resistance is equal to or greater than the weight, when motion becomes impossible. For a projectile the air through which it moves serves both to retard its motion and also to *maintain* its motion. The force which propels the object horizontally is transferred from the projector of the object to the air or other medium, which then transfers the force to the object through incremental layers of air until the force is eventually dissipated. A second explanation requires that the air in front of the projectile moves from in front of it to the back to fill the void produced by the object's motion. This rushing air then provides the force necessary for horizontal motion to continue until air resistance slows it and the object falls.[5]

Examination of the responses of the students to these questions reveals that they also believe that forces are needed to maintain motion, and that once the force pushing an object horizontally is removed, the horizontal motion quickly disappears and the object falls vertically. Also implicit in this reasoning is the failure to recognize that the horizontal and vertical components of velocity (to use modern terminology) are independent of one another.

The following examples illustrate the student's belief that "something" is necessary to maintain the horizontal speed of an object once it is released: "The bolt decreases in [horizontal] speed the second it leaves the ceiling." "Gravity pulls down; this is all that is acting on the ball." "The ball has lost its momentum and is pulled downward." "The force of the moving boat will make the ball fall behind the mast." "Once the ball is dropped there is no further force being applied in a forward direction."

Students who fail to recognize the independence of the horizontal and vertical components of velocity explain their answers in the following ways: "The bullet travels faster than the monkey falls." "The train is in motion forward and the bolt will only be affected by gravity." "Depends on the speed of the bullet." "The boat will be in motion horizontally while the ball is falling straight down vertically."

Several students indicated confusion between speed and acceleration. "The train is accelerating forward, and the bolt will have no forward acceleration." This confusion is occasionally compounded by students who fail to compare like quantities in two dimensions. "The ball has only verti-

Table IV. Responses of all students who mentioned air resistance in their reasons.

Answer	A	B	C	Statement: "Ignore air friction"	Change to "correct" answer
Question					
1	3	0	14	5	2
2	0	5	2	2	2
3	0	3	0	2	0
4	0	3	0	3	0
5	0	3	5	3	2
6	5	6	0	3	5

cal acceleration, no horizontal velocity or acceleration... ." "The bolt is accelerating vertically downward, but the car is moving horizontally with v."

Some students also indicate confusion regarding the time of fall of an object with the distance it is falling or with the speed it is traveling. For example: "The ball will fall a shorter distance to the ground, while the bullet will travel a certain distance forward." "Even though the bullet travels faster than the monkey, the monkey only had to move a little bit." "It takes time to reach the monkey. By that time, the monkey fell a distance." Several students replied correctly to the hunter–monkey problem, but their reasons were: "The monkey is slower than the bullet."

The reasons given by another group of students indicate incorrect use of technical terms or show confusion about the meaning of the term. "A ball has no horizontal velocity or acceleration or force... ." "The ball has the same momentum as the boat." "Because of gravity" was a reason given by several students to different questions.

Air resistance was an important factor in Aristotle's explanation of motion. It is also an effect which many students have experienced. The questions posed to the students in this study made no mention of air resistance, and students were not instructed to ignore it. Thus it is not unreasonable to expect that this might influence their responses. The reasons given by all of the students were studied to determine if they had made any assumptions about air resistance in their answers. Those questions and answers in which students mentioned air resistance in some way are shown in Table IV. Also indicated in the table is the number of students who specifically stated "ignore air friction" in their answer, and the number of "incorrect" answers which might be changed to "correct" if air resistance were taken into account.

Out of all of the students surveyed, 26 mentioned air resistance in their reasons for one or more questions. The largest number did so in selecting the correct answer to question 1. A common reason given was: "Since the size of the balls is the same, the air friction will be the same." Several students, who answered question 2 correctly, indicated that in the closed boxcar the air is moving at the same speed as the bolt and the hole.

The largest number of answers that would be changed to correct if air friction were taken into account appeared in question 6, where students indicated the wind would blow the ball behind the mast. (One student explained: "Wind resistance and gravity will cause the ball to fall behind the mast.") The majority of students, who indicated some possible role of air friction, explicitly or implicitly ignored its effect in their reason and selected the "correct" answer.

The relatively large number of students who mentioned air friction in some way in their answers indicates the importance of this factor in their answer selection. In the same way, failure of students to mention air resistance does not eliminate the possibility that they considered it. This also raises another question. If students are specifically instructed to ignore air friction, will they? The possible role that air friction plays in a student's response to these questions is an important one, and needs to be clarified in future research.

Comparing the combined answers of all students who had studied high school physics with those who had not reveals that a larger percentage of students with high school physics responded correctly on all six questions. However, the variation of the number of students who responded correctly to *similar* questions suggests that an understanding of the concepts involved may not be as firm as one might expect. The total number of students sampled in this study is fairly small, however, and an increased sample size would provide for a stronger inference.

A comparison of the percentages of students enrolled in the calculus-based course, whose mathematical skills would be expected to be greater than those with only a trigonometry background, did not do appreciably better in their responses. Indeed, for several questions the percentage of "correct" responses was less. One is led to hypothesize that the concepts involved in these questions are relatively independent of *mathematical* background. Again a larger sample would be desirable in order to resolve this question more fully.

The responses of these students to problems of trajectory motion reveal an intuitive understanding of motion that is different from that which is developed in textbooks. While some students considered the possible influence of air resistance in several answers, some of these same students and many of the others gave reasons for other answers in the ways discussed above. Students' views of the role of forces—both "impressed" and frictional—in explaining motion have been examined by Clement[6] and by McCloskey *et al.*[7] The responses of students in this study reveal a similar kind of reasoning.

Trowbridge and McDermott have documented the difficulties that beginning students have in understanding the concept of velocity in one dimension.[8] Their study indicates that students in introductory physics courses confuse the concepts of speed and position, and, that even after instruction, approximately one-fifth of the students in their study still confused these concepts. They also found that some students could provide a correct definition of velocity, but "...they did not understand the concept well enough to be able to determine a procedure they could use in a real physical situation for deciding if and when two objects have the same speed. Instead they fell back on the perceptually obvious phenomenon of passing."[9]

Trowbridge and McDermott also found that students display significant difficulty with the concept of acceleration in one dimension.[10] They found that students show confusion between position and acceleration, between velocity and acceleration, and that while some demonstrated understanding of the differences between velocity and change in velocity, they neglected the time interval over which the velocity change takes place. Even after instruction it was found that fewer than half of the students were able to apply the concept of acceleration accurately in a quantitative situation.[11]

The responses of students to the problems of trajectory motion reveal conceptual difficulties similar to those described by Trowbridge and McDermott. These responses also reveal an additional difficulty when one examines motion in two dimensions. Students not only must have a clear understanding of the differences between position, velocity, acceleration, and time of the position of the object, they must also recognize that the horizontal component of velocity of a projectile is independent of its vertical component. That it took until the time of Galileo before this problem was resolved indicates some of the difficulty of this concept.

INSTRUCTIONAL IMPLICATIONS

Explanation of the responses students give to problems in trajectory motion does not lie fully in the inherent difficulty of the involved concepts, however. While most students have had experience with the *phenomenon* of position, velocity, and acceleration (for example, in driving a car), the full meaning of these *concepts* has not yet been assimilated into their cognitive structures. Trowbridge and McDermott have noted "...that for some students the acquisition of physical concepts seems to depend strongly upon the establishment of satisfactory connections between these new concepts and the protoconcepts with which the student is already familiar."[12]

Renner has contrasted the differences in the thought processes of students who think only with information they have acquired through interaction with concrete objects, events, and/or situations (concrete operational thought) and the thought processes of students who are able to perform mental operations in a hypothetical-deductive manner (formal operational thought). He emphasizes that: "There is, however, a necessity for concrete operational thinkers to interact with concrete objects, events, and/or situations if they are to progress from the concrete to the formal thought stage."[13]

Arons, similarly, has commented on the approach of students to various physical situations. He observes that students who may correctly state Newton's first law fail to apply it correctly to an object propelled off the edge of a table in trajectory motion.

> These students can give a correct statement of the law of inertia, but they have no real understanding of its meaning. The idea is subtle; it takes time to assimilate from a sufficiently wide context of laboratory experience and thought experiments, but students are rarely afforded this luxury; all too frequently they are expected to be purged of their natural Aristotelianism simply by listening to a lecture, working a few sterile, end-of-chapter examples (not problems) that contribute neither insight nor understanding, and then regurgitating the verbal statement memorized as Newton's first law.[14]

The implications of this research are clear. As Renner has documented, many secondary school and college students still use concrete situations when approaching concepts such as those discussed above.[15] Similarly, students must be given the opportunity to examine and be aware of the ways in which they think about these concepts prior to the introduction of a more formal analysis of them. This was clearly the approach that Galileo, through the voice of Salviati, took in the *Dialogue*.

This author has used the questions reported in this research as an introduction to trajectory motion to students in a variety of introductory level courses. Students are encouraged to explain the reasons for their answers and to compare the differences in their choice with other class members. The variety of answers and reasons usually provides a spirited discussion and sets the rationale for examining the phenomenon in a direct way. Demonstrations and the filmloops of trajectory motion developed by Project Physics provide concrete illustrations of the trajectory of moving object.[16] These examples are then used to develop a description of trajectory motion consistent with the textbook analysis. This viewpoint and its reasons are now more readily accepted and understood since the inherent barriers to understanding had been confronted earlier.

Students often enter introductory courses in physics with concepts which may differ considerably from the concepts they will encounter in class and in the textbook. Their ways of thinking about these phenomena and concepts are also different from what we may expect or require. Confronting students with their misconceptions (or preconceptions) and allowing them the necessary experiences and time to resolve them will, hopefully, lead to a fuller understanding of these concepts and promote the student's overall intellectual progress as well.

ACKNOWLEDGMENTS

The author expresses his appreciation to his colleagues at Southwest Missouri State University who used this questionnaire in their classes. Particular thanks is given to Lillian McDermott of the University of Washington for sharing her research on one-dimensional kinematics.

APPENDIX

1. A 5-pound ball and a 50-pound ball are dropped from the roof of the building. Each ball is the same size, that is, it has the same diameter. The 50-pound ball will hit the ground:
   A. sooner than the 5-pound ball.
   B. later than the 5-pound ball.
   C. at the same time as the 5-pound ball.
2. A train is traveling along a smooth, straight track at a speed of 60 miles per hour. One of the boxcars has a small hole in the floor. Directly above the hole is a bolt in the ceiling. Suddenly the bolt comes loose and falls. The bolt will:
   A. hit the floor in front of the hole.
   B. fall through the hole.
   C. hit the floor in back of the hole.
3. A rifle is mounted so that its barrel is horizontal and is pointed across a long, flat field. A heavy ball is placed at exactly the same level as the muzzle of the rifle. The gun is fired and at the same time that the bullet leaves the end of the barrel, the ball begins to fall straight down. The ball will hit the ground:
   A. sooner than the bullet.
   B. at the same time as the bullet.
   C. later than the bullet.
4. A rifle is mounted so that its barrel is horizontal and is pointed straight at a monkey in a tree. The rifle is fired at the monkey and at the same time that the bullet leaves

the end of the barrel, the monkey drops from the tree toward the ground. The bullet will:
A. pass over the monkey's head.
B. hit the monkey.
C. pass below the monkey.

5. A rider on a galloping horse holds a heavy ball out to his side and drops it. The ball will hit the ground:
A. immediately below the point where it was dropped.
B. immediately below the point the horse and rider are when the ball reaches the ground.
C. some position other than A or B. Please specify.

6. A boat is sailing rapidly along the surface of a smooth lake. A heavy ball is dropped from the top of a high mast. The ball will fall and
A. strike the deck at the foot of the mast.
B. strike the deck behind the foot of the mast depending on the speed of the boat.
C. strike the deck in front of the mast depending on the speed of the boat.

[1]E. J. Dijksterhuis, *The Mechanization of the World Picture* (Oxford University, New York, 1969), p. 30.

[2]Galileo Galilei, *Dialogue Concerning the Two Chief World Systems*, translated by Stillman Drake (University of California, Berkeley, 1962).

[3]A. Franklin, Am. J. Phys. **44**, 529 (1976).

[4]Reference 2, p. 126.

[5]Reference 3, pp. 529–530.

[6]J. Clement, Am. J. Phys. **50**, 66 (1982).

[7]M. McCloskey, A. Caramazza, and B. Green, Science **210**, 1129 (5 December 1980).

[8]D. E. Trowbridge and L. C. McDermott, Am. J. Phys. **48**, 1020 (1980).

[9]Reference 8, p. 1027.

[10]D. E. Trowbridge and L. C. McDermott, Am. J. Phys. **49**, 242 (1981).

[11]Reference 10, p. 253.

[12]Reference 8, p. 1028.

[13]J. W. Renner, Am. J. Phys. **44**, 219 (1976).

[14]A. Arons, Am. J. Phys. **41**, 772 (1973).

[15]Reference 13, pp. 219–220. See also J. W. Renner *et al.*, *Research, Teaching and Learning with the Piaget Model* (University of Oklahoma, Norman, 1976); F. P. Collea *et al.*, *Workshop on Physics Teaching and the Development of Reasoning* (American Association of Physics Teachers, Stony Brook, NY, 1975); and *Cognitive Process Instruction: Research on Teaching Thinking Skills*, edited by J. Lochhead and J. Clement (Franklin Institute, Philadelphia, 1979).

[16]*Project Physics Handbook* (Holt, Rinehart and Winston, New York, 1975), pp. 1/59–1/62. The films are *Galilean Relativity—Ball Dropped from Mast of Ship, Galilean Relativity—Object Dropped from Aircraft*, and *Galilean Relativity—Projectile Fired Vertically.*

Maximum projectile range with drag and lift, with particular application to golf

Herman Erlichson

Department of Applied Sciences, The College of Staten Island, Staten Island, New York 10301

(Received 23 October 1981; accepted for publication 16 June 1982)

This paper explores the interesting problem of projectile motion without the vacuum idealization. Particular attention is paid to golf ball trajectories with and without lift. No lift trajectories with linear and quadratic drag are considered first. Then, trajectories with lift and linear drag are investigated. Projection angles for maximum range are determined for all these cases. Computer solutions are used throughout, with a Runge–Kutta routine used for all cases except for the well-known closed solution for the no lift, linear drag projectile.

I. INTRODUCTION

One of the interesting questions which is usually left unanswered in introductory or intermediate mechanics courses[1] is "What angle do you need for maximum projectile range if you are not in vacuum?" Virtually everybody has had the experience of hitting a golf ball, or a baseball, or throwing a football, and there is a natural curiosity about how to project a ball in air in order to achieve maximum range.

Three previous papers in this Journal have addressed the problem of the projection angle for maximum range for the shot put,[2] the discus,[3] and for round pebbles or stones.[4] In the case of the shot put the air resistance has little effect on the range, whereas in our analyses the drag is always a significant factor. In the case of the discus analysis, the author assumes quadratic drag and lift forces, whereas our analysis of the golf ball problem uses linear drag and lift forces. The discus paper also has the effect of wind as a major factor, in our paper we assume no wind. It is interesting to note that the optimum projection angles for the quadratic drag golf ball with no lift found in this paper (38° and 35°) are similar to the optimum discus projection angles found by Frohlich[3] (33° to 39°) for no wind conditions. The paper on throwing pebbles assumes a quadratic drag force and no lift.

The author's interest in these matters was all the greater because he has been an avid golfer for a long time. The large discrepancy between the approximately 11 deg of loft for the golf driver club and the 45 deg maximum range angle for a vacuum was the motivation to begin a study of the question of maximum projectile range in the presence of air resistance, with particular application to the flight of a golf ball.

Of course, the first question which should be addressed is the nature of the resisting, or drag force. There was evidence in the literature[5] that the drag force on a golf ball is linear. This case is well known and can be solved exactly if